Calculus of Variations is a branch of Mathematics whose theory and results have been, and are currently, widely used in many fields of Physics, Engineering, Biology and Economics. The name Calculus of Variations dates back to Euler who first, together with Lagrange, introduced a systematic methods to dealing with problems such as refraction laws in geometrical optics, optimal profile of bodies moving in a fluid, minimal surfaces etcetera. Their approach has been then refined by the work of many authors that developed further tools tailored for applications to a wider class of problems.

The aim of this course is to provide then the basic concepts and results (and possibly a flavour of a bit more) necessary to enter in this subject.

More in detail the course will be divided into three parts. In the first one I will present the main properties of Sobolev spaces viewed as natural setting of classical problems in Calculus of Variations (in a relaxed form). In this respect both the existence of properly summable weak derivatives and the characterization as closure of (the space of) regular functions with respect to an integral norm would be justified in view of application to classical minimum problems as Dirichlet one. More precisely this task will be accomplished by using the direct methods of Calculus of variations that will be presented in both the topological and metric setting, pointing out the gap in-between the two frameworks with suitable examples. In particular we will introduce the notion of (topological or sequential) lower semi-continuity, weak coercivity, lower semi-continuity envelopes, relaxation.

In the second part I will push forward this approach applying the direct methods to integral functional defined in Sobolev spaces. Various examples, more or less classical, will be treated in details in order to highlight different phenomena and drawbacks. As a by-product we are led to investigate necessary and sufficient conditions for the lower semi-continuity of integral functionals with respect to the weak topology in Sobolev spaces. This analysis will be carried on first in the scalar setting. In this framework the notion of convexity plays a relevant role so some notions of convex analysis will be recalled and then exploited for establishing the lower semi-continuity result. Note that the main proofs will be given for autonomous functionals, i.e. with integrands depending only on the gradient, providing precise statements for the general case.

While in the scalar case the notion of convexity is enough, the vectorial setting need a deeper analysis and more refined tools. Hence, in the third part of the course convexity will be replaced by a weaker notion named as quasi-convexity in case of integrands with real values. This notion will be proved to be sufficient to established the lower semi-continuity result. In order to deal with functionals in presence of additional constraints also the notion of poly-convexity will be quoted as well as rank-1 convexity. Suitable applications to models in Nonlinear elasticity will be provided.

Finally I have to say that no special background is required for following the course apart for standard prerequisites of integration theory and basic Analysis. During the course precise references about notions and results quoted will be provided.